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Riesz–Thorin Theorem and l_p-Stability of Nonlinear Time-Varying Discrete Systems

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The interpolation theorem due to Riesz–Thorin is used along with Hölder's and Young's inequalities to derive some new conditions, more general than those in the literature, for the l_p-stability ($1 \leqslant p \leqslant \infty$) of a class of nonlinear time-varying discrete systems represented by a time invariant linear discrete part $\mathfrak{G}$ in feedback with a discrete nonlinear time-varying gain $k(n)\varphi(\cdot)$. These stability conditions are expressed in terms of a general multiplier (causal + anticausal) function and global upper and lower bounds on the normalized rate of growth, $(k(n+1)/k(n))$, of the time-varying gain.

1. INTRODUCTION

We deal with the l_p-stability of a non-linear time-varying feedback discrete (or sampled data) system governed by the equation

$$\begin{aligned} v(n) &= x(n) - k(n)\varphi(y(n)) \\ y(n) &= (\mathfrak{G}v)(n) = \sum_{i=1}^{\infty} g_i v(n-\tau_i) + \sum_{i=0}^{\infty} g(i) v(n-i) \end{aligned} \tag{1}$$

for all $i \in I^+$, the set of nonnegative integers, where $\mathfrak{G}$ is a time invariant linear discrete operator, $\varphi(\cdot)$ is a memoryless, first and third quadrant, monotone nonlinearity, and $k(n)$ is a discrete time-varying gain; $x(\cdot)$, $v(\cdot)$, and $y(\cdot)$ are respectively the input sequence to the system, the error signal sequence, and output sequence of the system. For assumptions on the components of (1), see Section 2 below.

In the present paper, the Riesz–Thorin interpolation theorem [1] is used along with the Hölder, Young, and other l_p-norm inequalities, invoking the "energy balance" arguments as used in [2a], in order to derive sufficient conditions for the system (1) to be l_p-stable ($1 \leqslant p \leqslant \infty$).

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The problem of l_2-stability of nonlinear discrete systems has had a long history, starting with the "circle criterion" of Tsypkin [3]. Later Jury and Lee [4], Narendra and Cho [5], and others derived l_2- (or equivalent) stability conditions by weakening the constraint on the linear part of the system in exchange for narrowing the class of $\varphi(\cdot)$ (exclusive) or, in [5], $k(n)$. However, the constraint on $k(\cdot)$ in [5] is a point-by-point bound on $(k(n+1)/k(n))$, the normalized rate of growth of $k(\cdot)$. For *time invariant* systems, O'Shea and Younis [6] and Brockett and Willems [7] derived (asymptotic, l_2-) stability conditions in terms of a general causal + anti-causal multiplier function. See Davis [8] for stability results for linear systems with periodic feedback.

Motivated by the recent contribution of Mossaheb [9] for continuous-time systems, we derive more general l_p-stability $(1 \leqslant p \leqslant \infty)$ conditions using the Riesz–Thorin interpolation theorem. In particular, for $p=2$, the stability conditions are more general than the results of Narendra and Cho [5]. Moreover, the present approach enables us to deal with monotone nonlinearities too, unlike the double application of the circle criterion by Mossaheb [9].

The main results of the paper are Theorems 1 and 2 (Section 2) and the proofs (Appendix 4), which are based on Lemmas 1–7 (Section 3). It is believed that Lemmas 4, 6, and 7 are new and interesting by themselves. Two examples (Section 4) illustrate the use of the theorems.

In order to conserve space, we assume that the reader is familiar with the concepts of l_p, extended l_p-spaces, and l_p-stability. See, for instance, Willems [10, pp. 4–5, 12–13].

2. Some Preliminaries and Statement of the Main Results

Let x denote a real valued sequence $\{x(\cdot)\}$ on I^+. Let $l_p[0, \infty)$ be normed with the norm

$$\|x(\cdot)\|_p = \left(\sum_{i=0}^{\infty} |x(i)|^p \right)^{1/p}.$$

The system described by (1) is l_p-stable if, for $x \in l_p[0, \infty)$, an inequality of the type

$$\|v\|_p \leqslant \text{const} \|x\|_p$$

holds.

Concerning the system (1), we make the following assumptions:

(A1) $\{v(\cdot)\}$ and $\{y(\cdot)\}$ are in $l_{pe} \cap l_{qe}[0, \infty)$ with $p^{-1} + q^{-1} = 1$, $p \geqslant 1$;

(A2) $\{g(\cdot)\}$ is in $l_1[0, \infty)$, and there is a constant $\varepsilon_0 > 0$ such that $\{g(n) \exp(\varepsilon_0 n)\}$ is also in $l_1[0, \infty)$;

(A3) $\{g_i\}$ is a sequence in l_1; τ_i is a sequence in I^+;

(A4) $k(\cdot)$ takes values in $[\varepsilon, \infty)$ for some constant $\varepsilon > 0$;

(A5) $\varphi(\cdot)$ is a real valued function on $(-\infty, \infty)$; and $\varphi(0) = 0$.

There exist constants $q_1 > 0$, $q_2 > 0$ with $q_1 < q_2$ such that $q_1 \sigma^2 \leqslant \varphi(\sigma)\sigma \leqslant q_2 \sigma^2$ for all $\sigma \neq 0$.

(i) $\varphi(\cdot) \in C$, if $\sigma\varphi(\sigma) > 0$ for all $\sigma \neq 0$.

(ii) $\varphi(\cdot) \in C_m$, the class of monotone nondecreasing functions if $(\sigma_1 - \sigma_2)(\varphi(\sigma_1) - \varphi(\sigma_2)) \geqslant 0$ for all σ_1, σ_2.

Let $G(z)$ denote the z-transform of $\mathfrak{G}$, i.e.,

$$G(z) = \sum_{i=1}^{\infty} g_i z^{-\tau_i} + \sum_{i=0}^{\infty} g(i) z^{-i}. \tag{2}$$

THE l_p-STABILITY PROBLEM. Find conditions on $k(n)$ and $G(z)$ which ensure that v is in l_p with $\|v\|_p \leqslant \text{const.} \|x\|_p$.

A solution to the stability problem involves some terms and definitions.

DEFINITION 1. Let P denote the class of operators $\mathfrak{M}$: $l_{pe} \to l_{pe}$ satisfying an equation of the type

$$(Mx)(n) = x(n) + \sum_{i=1}^{\infty} m_i x(n - \sigma_i) + \sum_{i=1}^{\infty} m_i' x(n + \sigma_i') + \sum_{i=-\infty}^{\infty} m(i) x(n-i), \tag{3}$$

where the sequences $\{m_i\}$, $\{m_i'\}$ are in l_1, the sequences $\{\sigma_i\}$, $\{\sigma_i'\}$ are in I^+; $\{m(\cdot)\}$ is a real valued sequence on $(-\infty, \infty)$, and is in $l_1(-\infty, \infty)$.

The z-transform of $\mathfrak{M}$ is given by

$$M(z) = 1 + \sum_{i=1}^{\infty} m_i z^{-\sigma_i} + \sum_{i=1}^{\infty} m_i' z^{+\sigma_i'} + \sum_{i=-\infty}^{\infty} m(i) z^{-i}. \tag{4}$$

DEFINITION 2. Let $\mathfrak{K}$ be the class of real valued functions $k(\cdot)$ on I^+ with each $k(\cdot)$ having constants $\underline{k} > 0$ and $\bar{k} \geqslant \underline{k}$ for which $\underline{k} \leqslant k(n) \leqslant \bar{k}$ for all $n \in I^+$.

DEFINITION 3. With $k \in \mathfrak{K}$, let $\theta(n) = \log(k(n+1)/k(n))$;

$$\theta^+(n) = \begin{cases} \theta(n) & \text{for all} \quad \theta(n) > 0 \\ 0 & \text{for all} \quad \theta(n) \leqslant 0 \end{cases}$$

and

$$\theta^-(n) = \begin{cases} \theta(n) & \text{for all} \quad \theta(n) < 0 \\ 0 & \text{for all} \quad \theta(n) \geqslant 0 \end{cases}$$

Evidently $\theta(n) = \theta^+(n) + \theta^-(n)$.

Noting that $y(n)$ is a real valued function of $n \in I^+$, and that $\varphi(y)$ is a real valued continuous function of y, we require the following characteristics of $\varphi(\cdot)$ in Theorem 1:

DEFINITION 4. Let

$$\begin{aligned} \delta_s &= \sup_{y,\, y \neq 0} \left(\int_0^y \varphi(w)\, dw/\varphi(y)\, y \right); \\ \delta_i &= \inf_{y,\, y \neq 0} \left(\int_0^y \varphi(w)\, dw/\varphi(y)\, y \right). \end{aligned} \tag{5}$$

Note that for $\varphi(\cdot) \in C_m$, $0 \leqslant \delta_s \leqslant 1$.

Throughout the rest of the paper, Re denotes "the real part of" and $*$ denotes convolution.

The main results of the paper are the following two theorems:

THEOREM 1. *For $\varphi \in C_m$, if there exists an operator $\mathfrak{M}$ in P with $m(\cdot) \leqslant 0$, $m_i \leqslant 0$, and $m_i' \leqslant 0$ for all $i = 1, 2, \ldots$, such that*

(a) *for some positive constants ξ, ζ*

$$\begin{aligned} &\sum_{i=1}^{\infty} |m_i| \exp(\xi\sigma_i) + \sum_{i=1}^{\infty} |m_i'| \exp(\zeta\sigma_i') \\ &\qquad + \sum_{i=0}^{\infty} |m(i)| \exp(\xi i) + \sum_{i=-\infty}^{0} |m(i)| \exp(-\zeta i) \\ &\leqslant 1/(1 + \delta_s - \delta_i); \end{aligned} \tag{6}$$

(b)

$$\operatorname{Re} M(ze^{-\varepsilon})G(ze^{-\varepsilon}) \geqslant \delta > 0 \qquad \textit{for all} \quad |z| = 1, \tag{7}$$

and some positive constant ε (which in view of hypothesis (a) *and assumption* (A3) *is less than ε_0); and*

(c) *for some positive constants* K_1 *and* K_2, *and for all finite* $N>0$ *and all* $n_0 \geqslant 0$,

$$\frac{1}{N}\sum_{n_0}^{n_0+N} \theta^+(n) \leqslant K_1; \qquad \frac{1}{N}\sum_{n_0}^{n_0} \theta^-(n) \geqslant -K_2 \tag{8}$$

and, in addition, either

$$\begin{aligned} &\text{(i)} \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{n_0}^{n_0+N} \theta^+(n) \leqslant \xi \\ &\qquad\qquad \lim_{N\to\infty} \frac{1}{N}\sum_{n_0}^{n_0+N} \theta^-(n) \geqslant -\zeta \end{aligned} \tag{9}$$

or

(ii) *for* $\xi=\zeta$, $\theta(n)$ *is unrestricted,*

then the feedback system governed by (1) *is* l_p*-stable.*

THEOREM 2. *For* $\varphi \in C$, *if* $\operatorname{Re}(ze^{-\varepsilon}) \geqslant \delta > 0$ *for all* $|z|=1$, *and some positive constant* ε (*which in view of assumption* (A3) *is less than or equal to* ε_0), *then the feedback system governed by* (1) *is* l_p*-stable.*

3. PRINCIPAL LEMMAS

The reader is assumed to be familiar with the inequalities of Hölder and Young, with respect to norms of functions in l_p and l_q. See, for instance, Beckenbach and Bellman [11, pp. 19–23] and Reed and Simon [1, pp. 28–29].

As we are dealing with convolution of l_p and l_1 functions in (1), we need to employ the following lemma (see Stein and Weiss [12, p. 3]):

LEMMA 1. *Of* $f \in l_p$ *for* $1 \leqslant p \leqslant \infty$, *and* $g \in l_1$, *then* $h = f * g$ *satisfies the norm inequality*

$$\|h\|_p \leqslant \|f\|_p \|g\|_1.$$

COROLLARY 1. *If* $f \in l_p$ *for* $1 \leqslant p \leqslant \infty$ *and* $g_1, g_2 \in l_1$, *then* $h_1 = (f * g_1 * g_2)$ *satisfies the norm inequality*

$$\|h_1\|_p \leqslant \|f\|_p \|g_1\|_1 \|g_2\|_1.$$

The following lemma, which is believed to be new and interesting by itself, is required in the proof of the main results of the paper.

LEMMA 2. *If $f \in l_p \cap l_q$, then for some $\delta > 0$ independent of N, the following inequality holds*:

$$\sum_{i=1}^{N} |f(i)|^2 \geqslant \delta \left(\sum_{i=0}^{N} |f(i)|^p \right)^{1/2} \left(\sum_{i=0}^{N} |f(i)|^q \right)^{1/q}. \tag{10}$$

Proof. See Appendix 1.

On the basis of the assumptions (A1)–(A5), we get, in the course of proving the theorems, an inequality relating the $l_p \cap l_q$ norm of v with the $l_p \cap l_q$ norm of x. Thereafter, we arrive at the l_p-stability conditions for (1) by invoking the following Riesz–Thorin interpolation theorem:

LEMMA 3 (Riesz–Thorin). *Let $1 \leqslant p_0, p_1, q_1 \leqslant \infty$ and suppose Q is a linear transformation from $l_{p_0} \cap l_{p_1}$ to $l_{q_0} \cap l_{q_1}$ which satisfies the inequalities*

$$\|Qf\|_{q_0} \leqslant \alpha_0 \|f\|_{p_0}; \qquad \|Qf\|_{q_1} \leqslant \alpha_1 \|f\|_{p_1}$$

for some positive constants α_0 and α_1. Then for each $f \in l_{p_0} \cap l_{p_1}$ and each $s \in (0, 1)$, $Qf \in l_{q_s}$ and

$$\|Qf\|_{q_s} \leqslant \alpha_s \|f\|_{p_s},$$

where

$$p_s^{-1} = sp_1^{-1} + (1-s)\, p_0^{-1},$$

$$q_s^{-1} = sq_1^{-1} + (1-s)\, q_0^{-1},$$

and

$$\alpha_s = \alpha_0^{(1-s)} \cdot \alpha_1^s.$$

Proof. See Reed and Simon [1, pp. 27–28] for a result in a more generated context.

LEMMA 4. *If*

(a) *the operator $\mathfrak{M}$ belongs to P_0 with $m(\cdot) \leqslant 0$, $m_i \leqslant 0$ and $m_i' \leqslant 0$ for all $i = 1, 2, ...$;*

(b) *for some nonnegative constants ξ and ζ, inequality* (6) *is satisfied; and*

(c) *for these values of ξ and ζ, with $f(\cdot) \in \mathfrak{K}$, $f(n)k(n)\exp(-\xi n)$ is nonincreasing and $f(n)k(n)\exp(\zeta n)$ nondecreasing for all $n \geqslant 0$,*

then the following inequality holds:

$$\sum_{i=0}^{N} f(n)(\mathfrak{M}x)(n)k(n)\varphi(x(n)) \geqslant 0$$

for all x in the domain of $\mathfrak{M}$ and for all $N \geqslant 0$.

Proof. See Appendix 2.

The next lemma deals with the factorization of time-varying gains.

LEMMA 5. *If there exists a time multiplier function $f(\cdot)$ in $\mathfrak{R}$ satisfying hypothesis* (c) *of Lemma* 4 *for some positive constants ξ and ζ, then the hypothesis* (c) *of Theorem* 1 *holds.*

Proof. See Appendix 3.

Remark 1. Lemma 4 is believed to be a significant generalization of the results of O'Shea and Younis [6] and Brockett and Willems [7] as applied to nonlinear time-varying gains in cascade with a time invariant convolution operator.

Remark 2. Lemma 5 is believed to be novel as applied to discrete time-varying gains.

The proof of Theorem 1, which is based on the above lemmas, is given in Appendix 4. The proof of Theorem 2 is similar and hence is omitted.

Remark 3. The hypothesis of Theorem 2, when so worded as to account for finite feedback gain, is the same as the circle theorem hypothesis. No additional restrictions on g_i and $g(n)$ are required.

4. APPLICATION

We now illustrate the use of Theorem 1 by means of two examples.

EXAMPLE 1. Consider a discrete nonlinear time-varying feedback system with

$$G(z) = (z+0.9)(z+0.225)/(z-0.9)(z-0.225)$$

and $\varphi(\cdot) \in C_{mo}$ and $(\delta_s - \delta_i) = 0.1$.

If we choose, for simplicity, $M(z) = (z-0.225)/(z+0.225)$, with the convergence region in the complex z-plane given by $|z| > 0.225$, corresponding to a causal $M(z)$, then hypothesis (b) of Theorem 1 is satisfied for, say, $\varepsilon = 0.10$.

We have, for $|z| > 0.225$,

$$M(z) = (1-0.45)\, z^{-1} - 0.225(z^{-2}) + (0.225)^2\, z^{-3} - \cdots.$$

With ξ first chosen such that $\exp(\xi) < 4.44$, hypothesis (a) requires satisfaction of the following inequality for ξ,

$$\begin{aligned} 0.45 \sum_{i=0}^{\infty} &\exp(\xi)\{1 + 0.225\exp(\xi) + (0.225)^2 \exp(2\xi) + \cdots\} \\ &< 1/(1+\delta_s - \delta_i) = 0.909, \end{aligned}$$

from which we get

$$\exp(\xi) < 1.39$$

or

$$\xi < 0.329.$$

Therefore the system under consideration is l_p-stable if

(i) for some positive constants K_1 and K_2, and for all finite $N > 0$ and all $n_0 \geqslant 0$,

$$\frac{1}{N} \sum_{n_0}^{n_0+N} \theta^+(n) \leqslant K_1; \qquad \left(\frac{1}{N}\right) \sum_{n_0}^{n_0+N} \theta^-(n) \geqslant -K_2;$$

(ii) $\lim_{N \to \infty} (1/N) \sum_{n_0}^{n_0+N} \theta^+(n) \leqslant 0.329$.

Note that, for finite n, the restriction on $\theta^+(n)$ is mere boundedness.

In contrast, Narendra and Cho [5] assert asymptotic stability of the system if it is *linear* and

$$(k(n+1)/k(n)) < \beta^2 \qquad \text{for } all \quad n \geqslant 0,$$

where β is obtained from requiring that $M(z/\beta)$ should be passive. In this case, $M(z/\beta)$ is passive if $\beta < 4.44$.

EXAMPLE 2. Consider a nonlinear discrete time-varying system with

$$G(z) = (z^2 - 0.81)^2/(z^2 + 0.2025)^2 + (1/K_0)$$

(K_0 being large but finite) and $\varphi(\cdot) \in C_m$ with $\delta_s - \delta_i = 0.014$. A multiplier of the type considered by Narendra and Cho [5] cannot be chosen in view of large positive and negative excursions of the argument of $G(z)$, as ω varies from 0 to π.

If the multiplier is chosen as

$$M(z) = (z^2 - 1.5)/(z^2 - 8.5),$$

with the region of convergence given by $|z| < 2.9$ (corresponding to a non-causal function) it can be verified that hypothesis (b) of Theorem 1 is satisfied (for some K_0 large but finite). We have

$$M(z) = 1 - (0.8235)\{1 + (0.1176)z^2 + (0.1176)^2 z^4 + \cdots\}$$

with ζ first chosen such that $\exp(2\zeta) < 8.5$, and hypothesis (a) requires that the following inequality for ζ be satisfied,

$$(0.8235)\{1 + (0.1176)\exp(2\zeta) + (0.1176)^2 \exp(4\zeta) + \cdots\} < 1/(1.014),$$

from which we get

$$\exp(2\zeta) < 1.402$$

or

$$\zeta < 0.1689.$$

Therefore the system under consideration is l_p-stable if

(i) for some positive constants K_1 and K_2,

$$\left(\frac{1}{N}\right)\sum_{n_0}^{n_0+N} \theta^+(n) \leqslant K_1; \qquad \left(\frac{1}{N}\right)\sum_{n_0}^{n_0+N} \theta^-(n) \geqslant -K_2;$$

(ii) $\lim_{N\to\infty}(1/N)\sum_{n_0}^{n_0+N}\theta^-(n) \geqslant -0.1689$.

Remark 4. The result of Mossaheb [9] (when specialized to the discrete case) cannot be applied to the two examples (for the same range of K_0, where applicable, considered here) in view of the variation of the argument of $G(z)$ as ω varies from 0 to π.

5. Conclusions

In the framework normally used for deriving l_2-stability conditions, Hölder's inequality and the interpolation theorem of Riesz–Thorin are invoked to establish l_p-stability ($1 \leqslant p \leqslant \infty$) conditions for a class of nonlinear time-varying feedback discrete systems governed by (1). The method of proof seems to be novel. The stability conditions are believed to be more general and less restrictive than the available results in the literature.

APPENDIX 1: Proof of Lemma 4

Consider the discrete inequality

$$\sum_{i=1}^{n} |f_i|^2 \geqslant \delta \left\{\sum_{i=1}^{n} |f_i|^p\right\}^{(1/p)} \times \left\{\sum_{i=1}^{n} |f_i|^q\right\}^{(1/q)} \quad \text{for all} \quad n \geqslant 0, \tag{10}$$

for some $\delta > 0$ independent of n, and nonnegative p, q satisfying $(p^{-1}+q^{-1}) = 1$, which we propose to establish by mathematical induction.

Let

$$S_n^{(2)} \triangleq \sum_{i=1}^{n} |f_i|^2;$$

$$S_n^{(p)} \triangleq \left\{\sum_{i=1}^{n} |f_i|^p\right\}^{(1/p)} \tag{11}$$

$$S_n^{(q)} \triangleq \left\{\sum_{i=1}^{n} |f_i|^q\right\}^{(1/q)}.$$

The inequality (10) is valid for $n=1$:

$$|f_1|^2 \geqslant \delta\{|f_1|^p\}^{(1/p)}\{|f_1|^q\}^{(1/q)} \qquad \text{for some} \quad \delta \in (0, 1].$$

Assume now that (10) is valid for n replaced by $(n-1)$. That is,

$$S_{n-1}^{(2)} \geqslant \delta\, S_{n-1}^{(p)}\, S_{n-1}^{(q)} \qquad \text{for some} \quad \delta > 0. \tag{12}$$

Add $|f_n|^2$ to both the sides of (12) to get

$$\begin{aligned} S_n^{(2)} \geqslant \delta\, S_{n-1}^{(p)} S_{n-1}^{(q)} &+ (|f_n|^p)^{(1/p)}\,(|f_n|^q)^{(1/q)} \\ &+ (1-\delta)|f_n|^2, \qquad \text{for some} \quad \delta > 0. \end{aligned} \tag{13}$$

Let, further,

$$\alpha_i \triangleq |f_i|^p; \qquad \beta_i \triangleq |f_i|^q.$$

Then from (13), we have

$$S_n^{(2)} \geqslant \delta\{S_{n-1}^{(p)} S_{n-1}^{(q)} + \alpha_n^{(1/p)}\,\beta_n^{(1/q)}\} + (1-\delta)\,|f_n|^2. \tag{14}$$

Now let

$$A \triangleq S_{n-1}^{(p)} S_{n-1}^{(q)} + \alpha_n^{(1/p)}\beta_n^{(1/q)}$$

and

$$B \triangleq S_n^{(p)} S_n^{(q)}.$$

Then, it can be verified that

$$\begin{aligned} B &= (A - \alpha_n^{(1/n)}\beta_n^{(1/q)})\left\{1 + \left(\alpha_n \Big/ \left(\sum_{i=1}^{n-1} \alpha_i\right)\right)\right\}^{(1/p)} \\ &\quad \times \left\{1 + \left(\beta_n \Big/ \left(\sum_{i=1}^{n-1} \beta_i\right)\right)\right\}^{(1/q)} \\ A &= \alpha_n^{(1/p)}\beta_n^{(1/q)} + \left(B \Big/ \left(1 + \left(\alpha_n \Big/ \left(\sum_{i=1}^{n-1} \alpha_i\right)\right)\right)^{(1/p)}\right. \\ &\quad \left. \times \left(1 + \left(\beta_n \Big/ \left(\sum_{i=1}^{n-1} \beta_i\right)\right)\right)^{(1/q)}\right). \end{aligned} \tag{15}$$

Noting that α_i's and β_i's are nonnegative for all $i \in I^+$, we get from (15) the inequality

$$A \geqslant \delta_1 B \qquad \text{for some} \quad \delta_1 > 0 \text{ independent of } n. \tag{16}$$

Hence, from (14) and (16), we have, for some $\delta, \delta_1 > 0$,

$$\begin{aligned} S_n^{(2)} &\geqslant \delta\, \delta_1\, S_n^{(p)} S_n^{(q)} + (1-\delta)|f_n|^2 \\ &\geqslant \delta_2\, S_n^{(p)} S_n^{(q)}, \qquad \text{for some} \quad \delta_2 \triangleq \delta\, \delta_1 > 0, \end{aligned}$$

independent of n.

Hence the lemma is proved.

APPENDIX 2: Proof of Lemma 4

We have

$$\begin{aligned} S_2 &= \sum_{i=0}^{N} f(i)k(i)\varphi(y(i))(\mathfrak{M}y)(i) \\ &= \sum_{i=0}^{N} f(i)k(i) \left\{ y(i) + \sum_{h=1}^{\infty} m_h\, y(i-\sigma_n) \right. \\ &\quad + \sum_{h=0}^{\infty} m(h)\, y(i-h) + \sum_{h=1}^{\infty} m'_h\, y(i+\sigma'_n) \\ &\quad \left. + \sum_{h=-\infty}^{0} m(h)\, y(i-h) \right\} \varphi(y(i)) \\ &= \sum_{i=0}^{N} f(i)k(i)\, e^{-\xi i} \left\{ \frac{1}{2}\, y(i)\, e^{\xi i} + \sum_{h=1}^{\infty} m_h e^{\xi i} y(i-\sigma_h) \right. \\ &\quad \left. + \sum_{h=0}^{\infty} m(h)\, e^{\xi i} y(i-h) \right\} \varphi(y(i) + \sum_{i=0}^{N} f(i)k(i)\, e^{\zeta i} \left\{ \frac{1}{2}\, y(i)\, e^{-\zeta i} \right. \\ &\quad \left. + \sum_{h=1}^{\infty} m'_h e^{-\zeta i} y(i+\sigma'_h) + \sum_{h=-\infty}^{0} m(h)\, e^{-\zeta i} y(i-h) \right\} \varphi(y(i)), \end{aligned} \tag{17}$$

where ξ, ζ are nonnegative constants.

Since $f(i)k(i)\, e^{-\xi i}$ is nonincreasing, by the mean value theorem, there is a number N' in $[0, N]$ for which the first summation S_3 (with respect to i) on the right-hand side of (17) becomes

$$\begin{aligned} S_3 = f(0)k(0) \sum_{i=0}^{N'} &\left\{ \frac{1}{2}\, y(i)\, e^{\xi i} + \sum_{h=1}^{\infty} m_h e^{\xi i} y(i-\sigma_h) \right. \\ &\left. \sum_{h=0}^{\infty} m(h)\, e^{\xi i} y(i-h) \right\} \varphi(y(i)), \end{aligned}$$

which on interchange of the summation operators assumes the form

$$S_3 = f(0)k(0)\left[\left\{\sum_{i=0}^{N'} \frac{1}{2}\, y(i)\varphi(y(i)\, e^{\xi i}\right\}\right.$$
$$+ \sum_{h=1}^{\infty} m_h e^{\xi\sigma_h} \sum_{i=0}^{N'} e^{\xi(i-\xi\sigma_h)}$$
$$\times\, y(i-\sigma_h)\varphi(y(i)) + \sum_{h=0}^{\infty} m(h)\, e^{\xi h}$$
$$\left.\times \sum_{i=0}^{N'} e^{\xi(i-h)} y(i-h)\, \varphi(y(i))\right]. \tag{18}$$

By virtue of Lemma 2 of [2b] it can beshown that the expression (18) is nonnegative if

$$\sum_{h=1}^{\infty} |m_h|\, e^{\xi\sigma_h} + \sum_{h=0} |m(h)|\, e^{\xi h} \leqslant 1/(2(1+\delta_s-\delta_i)).$$

Regarding the second summation S_4 (with respect to i) on the right-hand side of (17), we note that $f(i)k(i)\, e^{\zeta i}$ is given as nondecreasing. Hence by the (discrete version of the) second mean value theorem, there is a number N'' in $[0, N]$ such that

$$S_4 = f(N)k(N)e^{\zeta N}\left\{\sum_{i=N''}^{N} \frac{1}{2}\, y(i)\, e^{-\zeta i}\right.$$
$$+ \sum_{h=i}^{\infty} m'_h e^{\zeta\sigma'_h} y(i+\sigma'_h)\, e^{-\zeta(i+\sigma'_h)}$$
$$\left.+ \sum_{h=-\infty}^{0} m(h)\, e^{-\zeta h} y(i-h)\, e^{-\zeta(i-h)}\right\} \varphi(y(i)). \tag{19}$$

Proceeding in the manner given for the first summation S_3 on the right-hand side of (17), and using Lemma 2 of [2b], we conclude that S_4 is nonnegative if

$$\sum_{h=i}^{\infty} |m'_h|\, e^{\zeta\sigma'_h} + \sum_{h=-\infty}^{0} |m(h)|\, e^{-\zeta h} \leqslant 1/(2(1+\delta_s-\delta_i)).$$

The lemma is proved.

APPENDIX 3: PROOF OF LEMMA 5

We have

$$f(i+1)k(i+1)\, e^{-\xi(i+1)} \leqslant f(i)k(i)\, e^{-\xi i}, \qquad i \in I^+ \tag{20}$$

and

$$f(i+1)k(i+1)\,e^{\zeta(i+1)} \geqslant f(i)k(i)\,e^{\zeta i}, \qquad i \in I^+ \tag{21}$$

or combining (20) and (21),

$$e^{-\zeta} \leqslant \frac{f(i+1)k(i+1)}{f(i)k(i)} \leqslant e^{\xi}, \qquad i \in I^+,$$

from which we conclude that

$$-\zeta \leqslant \log\left(\frac{f(i+1)}{f(i)}\right) + \log\left(\frac{k(i+1)}{k(i)}\right) \leqslant \xi, \qquad i \in I^+. \tag{22}$$

If we choose

$$\log\left\{\frac{f(i+1)}{f(i)}\right\} = \{\xi - \theta^+(i)\} + \{-\zeta - \theta^-(i)\}, \qquad i \in I^+ \tag{23}$$

then inequality (22) is satisfied. Hence from (23), for all $n_0 \in I^+$, $n_0 \leqslant n \leqslant I^+$,

$$\begin{aligned} f(n) = f(n_0) \exp\{&\xi(n-n_0) - \theta^+(n_0) - \theta^+(n_0+1) \\ &- \cdots - \theta^+(n-1) - \zeta(n-n_0) - \theta^-(n_0) - \theta^-(n_0+1) \\ &- \cdots - \theta^-(n-1)\}. \end{aligned}$$

But $f(\cdot)$ is in $\mathfrak{K}$ and hence, for some positive constants β_1 and β_2 with $\beta_1 < \beta_2$,

$$\begin{aligned} \beta_1 \leqslant \exp\{&\xi(n-n_0) - \theta^+(n_0+1) - \cdots - \theta^+(n-1) - \theta^+(n_0) \\ &- \zeta(n-n_0) - \theta^-(n_0) - \theta^-(n_0+1) - \cdots - \theta^-(n-1)\} \\ \leqslant \beta_2, &\qquad \text{for all} \quad n_0 \in I^+,\ n_0 \leqslant n \in I^+. \end{aligned} \tag{24}$$

Inequality (24) can be satisfied in, for instance, the following two ways for some positive constants, $\gamma_1, \gamma_2, \gamma_3$, and γ_4 and for all $n_0 \in I^+$, $n_0 \leqslant n \in I^+$.

Case 1.

$$-\infty < -\gamma_1 \leqslant (\xi - \zeta)(n-n_0) - \theta(n_0) - \theta(n_0+1) - \cdots - \theta(n-1) \leqslant \gamma_2 < \infty. \tag{25}$$

Case 2.

$$-\infty < -\gamma_1 \leqslant -\zeta(n-n_0) - \theta^-(n_0) - \theta^-(n_0+1) - \cdots - \theta^-(n-1) \leqslant \gamma_2 < \infty; \tag{26}$$

$$-\infty < -\gamma_3 \leqslant \xi(n-n_0) - \theta^+(n_0) - \theta^+(n_0+1) - \cdots - \theta^+(n-1) \leqslant \gamma_4 < \infty. \tag{27}$$

Case 1. Inequality (25) can be reduced to

$$e^{-\gamma_2+(\xi-\zeta)(n-n_0)} \leqslant e^{\theta(n_0)+\theta(n_0+1)+\cdots+\theta(n-1)} \leqslant e^{-\gamma_1+(\xi-\zeta)(n-n_0)}$$

or

$$e^{-\gamma_2+(\xi-\zeta)(n-n_0)} \leqslant \frac{k(n_0+1)k(n_0+2)\cdots k(n)}{k(n_0)k(n_0+1)\cdots k(n+1)} \leqslant e^{-\gamma_1+(\xi-\zeta)(n-n_0)} \tag{28}$$

for all $n_0 \in I^+$, $n_0 \leqslant n \in I^+$. When $\xi \neq \zeta$, we conclude from (28) that $k(\cdot) \notin \mathfrak{K}$. Hence Case 1 is ruled out for $\xi \neq \zeta$. But when $\xi = \zeta$, inequality (28) merely implies that $k(\cdot) \in \mathfrak{K}$, and hence no restriction on $\theta(n)$ is imposed.

Case 2. From inequalities (26) and (27), respectively,

$$-\gamma_2-\zeta(n-n_0) \leqslant \theta^-(n_0)+\theta^-(n_0+1)+\cdots+\theta^-(n-1) \leqslant \gamma_1-\zeta(n-n_0) \tag{29}$$

$$-\gamma_4+\xi(n-n_0) \leqslant \theta^+(n_0)+\theta^+(n_0+1)+\cdots+\theta^+(n-1) \leqslant \gamma_3+\xi(n-n_0) \tag{30}$$

for all $n_0 \in I^+$, $n_0 \leqslant n \in I^+$. Hence by requiring that

$$|\theta^-(n_0)|+|\theta^-(n_0+1)|+\cdots+\cdots+|\theta^-(n-1)|,$$
$$\theta^+(n_0)+\theta^+(n_0+1)+\cdots+\cdots+\theta^+(n-1)$$

be bounded for all finite $n \geqslant n_0$ and

$$\operatorname*{Lim}_{(n-n_0)\to\infty} \left\{\frac{\theta^+(n_0)+\theta^+(n_0+1)+\cdots+\theta^+(n-1)}{(n-n_0)}\right\} \leqslant \xi$$

and

$$\operatorname*{Lim}_{(n-n_0)\to\infty} \left\{\frac{\theta^-(n_0)+\theta^-(n_0+1)+\cdots+\theta^-(n-1)}{(n-n_0)}\right\} \geqslant -\zeta,$$

inequalities (29) and (30) are satisfied. The lemma is proved.

APPENDIX 4: Proof of Theorem 1

Consider the summation, for any $N \geqslant 0$,

$$\rho(N) = \sum_{i=0}^{N} f(i)x(i)(\mathfrak{M}\mathfrak{G}v)(i), \tag{31}$$

which when we use (1) becomes

$$\rho(N) = \sum_{i=0}^{N} f(i) v(i)(\mathfrak{M}\mathfrak{G}v)(i) + \sum_{i=0}^{N} f(i) k(i) \varphi(y(i))(\mathfrak{M}y)(i). \tag{32}$$

Let the first summation on the right-hand side of (32) be denoted by S_1, and the second by S_2. For some $\varepsilon > 0$, we have

$$S_1 = \sum_{i=0}^{N} f(i) \exp(-2\varepsilon i) v(i)(\mathfrak{M}\mathfrak{G}v)(i) \exp(2\varepsilon i).$$

Suppose we choose $f \in \mathfrak{K}$ so that $f(i) \exp(-2\varepsilon i)$ in nonincreasing and $\varepsilon < \varepsilon_0$ (see hypothesis (b) of Theorem 1). Then we can invoke (the discrete version of) the mean value theorem. According to this, there is a number N' in $[0, N]$ for which

$$S_1 = f(0) \sum_{i=0}^{N'} v(i)(\mathfrak{M}\mathfrak{G}v)(i) \exp(2\varepsilon i),$$

where $f(0) > 0$ in view of our choice of $f \in \mathfrak{K}$. By hypothesis (b) of Theorem 1, $\operatorname{Re} M(ze^{-\varepsilon}) G(ze^{-\varepsilon}) \geqslant \delta > 0$ for all $|z| = 1$. Hence by (the discrete version of) Parseval's theorem,

$$S_1 \geqslant \delta_1 \|v_N\|^2$$

for some constant $\delta_1 > 0$, independent of N.

Invoking Lemma 4, we conclude that

$$S_1 \geqslant \delta_1 \|v_N\|^2 \geqslant \delta_2 \|v_N\|_p \|v_n\|_q \tag{33}$$

for some constant $\delta_2 > 0$, independent of N.

The second summation (S_2) on the right-hand side of (32) is nonnegative by virtue of Lemmas 6 and 7 if hypotheses (a) and (c) of the theorem statement are satisfied. Consequently, from (33) and (32), and from an application of Lemmas 1–3 to the summation of (31), we have the following two possibilities:

Possibility 1. Assume $x \in l_p$ and $v \in l_{q^e}$. Then

$$\delta_2 \|v_N\|_p \|v_N\|_q \leqslant \rho(N) \leqslant \bar{f}\beta \|x_N\|_p \|v_N\|_q, \tag{34}$$

where

$$\beta = \left\{1 + \sum_{i=1}^{\infty} (|m_i| + |m_i'|) + \sum_{i=-\infty}^{\infty} |m(i)|\right\}\left\{\sum_{i=1}^{\infty} |g_i| + \sum_{i=0}^{\infty} |g(i)|\right\}. \tag{35}$$

Possibility 2. Assume $x \in l_q$ and $v \in l_{pe}$. Then

$$\delta_2 \|v_N\|_p \|v_N\|_q \leqslant \rho(N) \leqslant \bar{f}\beta \|x_N\|_q \|v_N\|_p, \tag{36}$$

where β is as defined in (35). Hence, with constant $A = (\bar{f}\beta)/\delta_2$, we get, from (34) and (36), the inequalities

$$\|v_N\|_p \leqslant A\|x_N\|_p \quad \text{and} \quad \|v_N\|_q \leqslant A\|x_N\|_q, \tag{37}$$

which are valid for all $N > 0$.

We now propose to employ Lemma 5, and to this end identify the parameters

$$q_0 = \infty,\ p_0 = \infty;\ q_1 = 1,\ p_1 = 1;\ \alpha_0 = \alpha_1 = A.$$

We now follow the argument of Mossaheb [9, Sect. 4], and on the basis of the local square summability of any solution of (1), let y be any arbitrary solution of (1). Then define the equivalent linear gain $k_{\text{eq}}(n)$:

$$\begin{aligned} k_{\text{eq}}(n) &= k(n)\varphi(y(n))/(y(n)) \qquad &&\text{if} \quad y(n) \neq 0 \\ &= 0 &&\text{if} \quad y(n) = 0. \end{aligned}$$

Hence the feedback system (1) is equivalently governed by

$$\begin{aligned} v(n) &= x(n) - k_{\text{eq}}(n)\, y(n) \\ y(n) &= (\mathfrak{G}v)(n) = \sum_{i=1}^{\infty} g_i v(n - \tau_i) + \sum_{i=0}^{\infty} g(i) v(i - n) \end{aligned} \tag{38}$$

for all $n \geqslant 0$. As a consequence, we can consider the transformation of $x(\cdot)$ to $v(\cdot)$ to be linear in accordance with (38), and employ Lemma 5 to get

$$\|v_N\|_{q_s} \leqslant \alpha_s \|x_N\|_{p_s},$$

where $q_s = (1/s) = p_s$ with $s \in (0, 1)$ and $\alpha_s = A$. But we already have, from (37),

$$\|v_N\|_\infty \leqslant A\|x_N\|_\infty \quad \text{and} \quad \|v_N\|_1 \leqslant A\|x_N\|_1.$$

Therefore, we finally have

$$\|v_N\|_p \leqslant A\|x_N\|_p, \qquad 1 \leqslant p \leqslant \infty,$$

which is valid for all $N > 0$. The theorem is proved.

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